EĞRİSEL İNTEGRALLER

tanım :

$x = r_1(t)$, $y = r_2(t)$, $a \leq t \leq b$ ile parametrik denklemi

$r(t) = r_1(t)i + r_2(t)j$ ile vektörel denklemi

verilen eğri için yay uzunluğuna göre eğrisel integrali

$$\int_C f(x,y)\,ds = \int_C f(r(t))\,ds$$ olarak tanımlanır.

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(r_1(t), r_2(t))\sqrt{[r_1'(t)]^2 + [r_2'(t)]^2}\,dt}$$

UYGULAMA

① $\int_C (x^2-y^2)\,ds$ $\quad C : r(t) = a\cos t\, i + a\sin t\, j$, $0 \leq t \leq \frac{\pi}{4}$ verilsin.

$r'(t) = -a\sin t\, i + a\cos t\, j \Rightarrow \|r'(t)\| = a$

$$\int_C (x^2-y^2)\,ds = \int_0^{\pi/4} (a^2\cos^2 t - a^2\sin^2 t)\,a\,dt$$

$$= a^3 \int_0^{\pi/4} \cos 2t\,dt = \frac{a^3}{2}$$

② $\int_C 2xy\,dx + x^2 dy$ $\quad$ C: $y = x^{1/2}$ $\quad 0 \le x \le 1$

öyleyse $\begin{matrix} x = t \\ y = t^{1/2} \end{matrix}$) o.ü $\quad r(t) = t\,i + t^{1/2} j \quad 0 \le t \le 1$ dır.

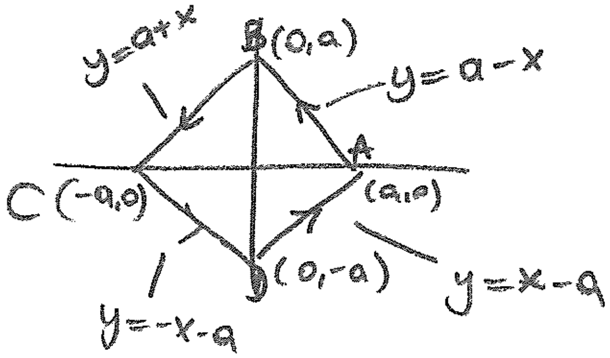
$$I = \int_0^1 \left(2.t.t^{1/2} + t^2 \frac{1}{2} t^{-1/2}\right) dt = 1$$ olarak bulunur

③ C eğrisi $A(0,0,0)$ noktasını $B(3,4,7)$ noktasına birleştiren doğru parçası ise $\int_C (2x+3y+z)\,ds = ?$

C: $\left.\begin{matrix} x = 3t \\ y = 4t \\ z = 7t \end{matrix}\right\} t \in [0,1] \Rightarrow \begin{matrix} r(t) = 3t\,i + 4t\,j + 7t\,k \\ \|r'(t)\| = \sqrt{74} \end{matrix}$

$$\int_C (2x+3y+z)\,ds = \int_{t=0}^{1} (2.3t + 3.4t + 7t)\sqrt{74}\,dt = \frac{25\sqrt{74}}{8}$$

④ C eğrisi $|x|+|y|=a$ karesi old. göre $\int_C xy\,ds = ?$



$$\int_C xy\,ds = \int_{AB} xy\,ds + \int_{BC} xy\,ds + \int_{CD} xy\,ds + \int_{DA} xy$$

$ds = \sqrt{1+(g')^2}\,dx \quad (g(x)=y)$

$$\int_{AB} xy\,ds = \int_0^a x(a-x)\,\sqrt{2}\,dx = -\frac{\sqrt{2}}{6}a^3$$

$$\int_{BC} xy\,ds = \int_0^{-a} x(a+x)\sqrt{2}\,dx = \frac{\sqrt{2}}{6}a^3$$

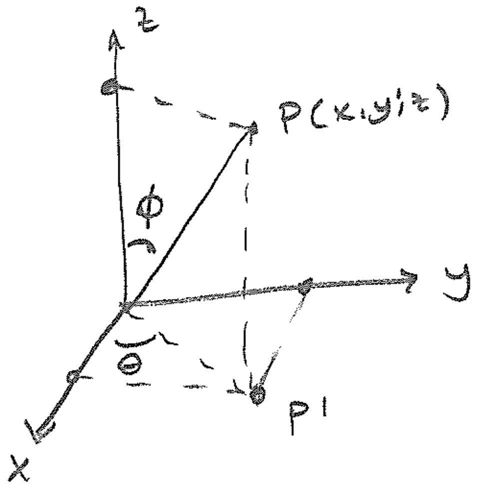
$$\int_{CD} xy\,ds = \frac{\sqrt{2}}{6}a^3 \qquad \int_{DA} x.y\,ds = -\frac{\sqrt{2}}{6}a^3$$

$$\Rightarrow \int_C xy\,ds = 0 \;//$$

ÜÇ KATLI İNTEGRALLER

A) Küresel Koor.

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$



$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \rho^2 \sin\phi$$

B) Silindirik Koor.

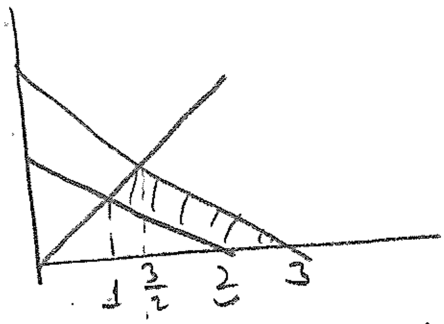
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$$

\# Dönüşümlerde Jakobiyen matrisi unutma!

$$\iiint_G f(x,y,z)\,dx\,dy\,dz = \iiint_D f(u,v,w) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

① G bölgesi $z=0$, $y=0$, $y=x$, $x+y=2$, $x+y+z=3$ düzlemleri tarafından sınırlanan bölge o.ü $f(x,y,z)=x$ ile verilen fonk. bu bölge üzerindeki üç katlı integralini hesaplayınız.

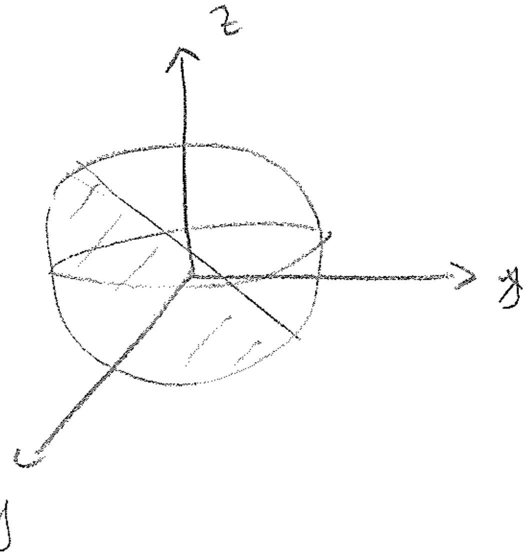


$$I=\int_{1}^{\frac{3}{2}}\int_{2-x}^{x}\int_{0}^{3-x-y} x\,dz\,dy\,dx+\int_{\frac{3}{2}}^{2}\int_{2-x}^{3-x}\int_{0}^{3-x-y} x\,dz\,dy\,dx+\int_{2}^{3}\int_{0}^{3-x}\int_{0}^{3-x-y} x\,dz\,dy\,dx$$

$$=\frac{1}{3}$$

② $x=0$ ve $x=\sqrt{a^2-y^2-z^2}$ yüzeyleri tarafından sınırlandırılan bölgede $f=x$ fonk integrali?

$$\iiint_D x\,dz\,dy\,dx=\int_0^{\pi/2}\int_0^{\pi}\int_0^{a} \rho\sin\phi\cos\theta\;\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$



$x=\rho\sin\phi\cos\theta$

$y=\rho\sin\phi\sin\theta$

$z=\rho\cos\phi$

$x^2+y^2+z^2=a^2 \Rightarrow \rho=a$

③ G bölgesi üstten $x^2+y^2+z^2=2az$ küresi ve altta, $z=\sqrt{x^2+y^2}$ konisi ile sınırlanan bölge o.ü $\iiint_G dx\,dy\,dz = ?$

$$I=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2a\cos\phi} \rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

$x=\rho\sin\phi\cos\theta$

$y=\rho\sin\phi\sin\theta$

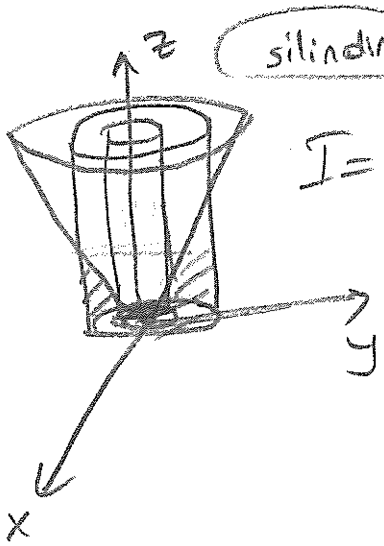
$z=\rho\cos\phi$

$x^2+y^2+z^2=2az \Rightarrow \rho^2=2a.\rho\cos\phi \Rightarrow \rho=2a\cos\phi$

$z^2+z^2=x^2+y^2+z^2 \Rightarrow 2\rho^2\cos^2\phi=\rho^2 \Rightarrow \rho=0$

$\Rightarrow \cos\phi=\frac{1}{\sqrt{2}} \Rightarrow \phi=\frac{\pi}{4}$

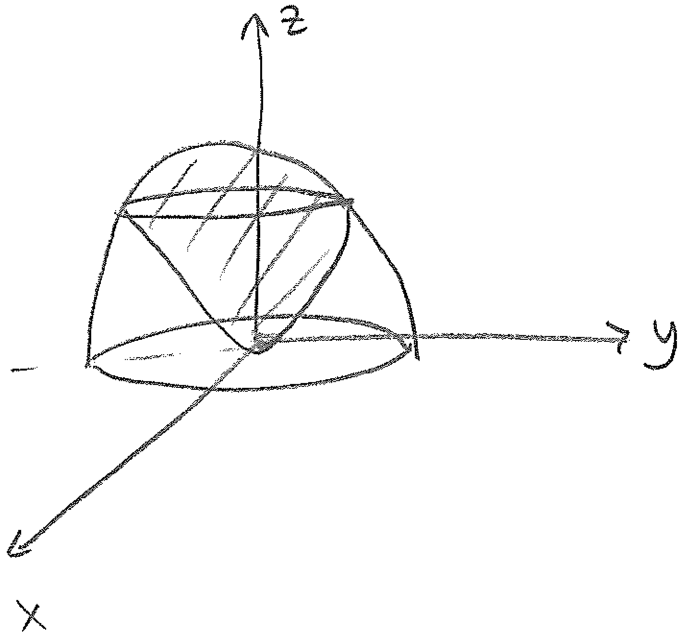
④ $x^2+y^2=4$, $x^2+y^2=1$ silindirleri ile $z^2=x^2+y^2$ konisi arasındaki bölgenin hacmi

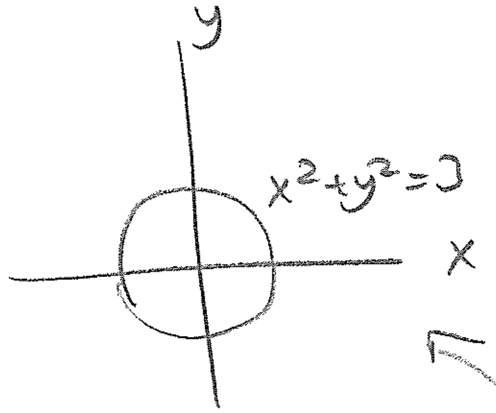


silindirik

$$I=2\int_0^{2\pi}\int_1^{2}\int_0^{\sqrt{x^2+y^2}} r\,dz\,dr\,d\theta$$

② Üstten $x^2+y^2+z^2=4$, altten $3z=x^2+y^2$ ile sınırlanan bölgenin hacmi?





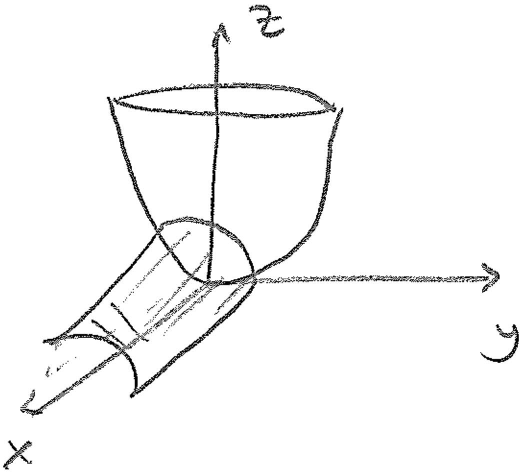
$$\underbrace{x^2+y^2}+z^2=4 \Rightarrow z^2+3z-4=0 \Rightarrow z=-4, \quad z=1$$

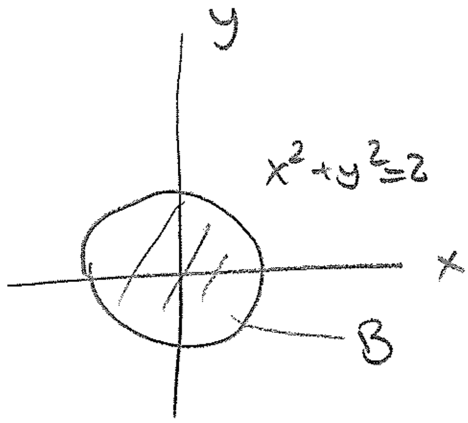
izdüşüm bölgesi

$$V=\iiint_{z=\frac{x^2+y^2}{3}}^{\sqrt{4-x^2-y^2}} dz\,dy\,dx = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_{z=\frac{x^2+y^2}{3}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$

ÜÇ KATLI İNTEGRALLER

① $z = 2x^2 + y^2$ paraboloidi ile $z = 4 - y^2$ silindiri tarafından sınırlanan bölgenin hacmi?



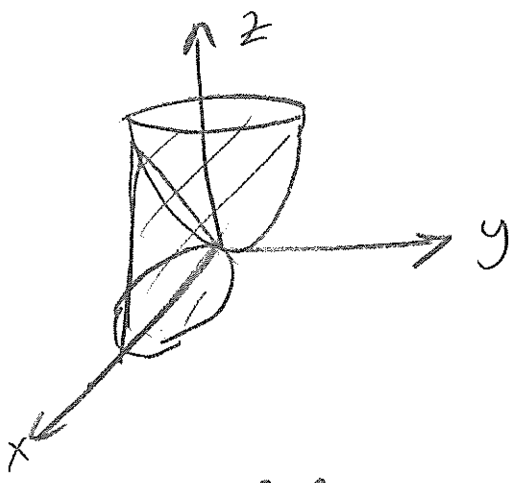


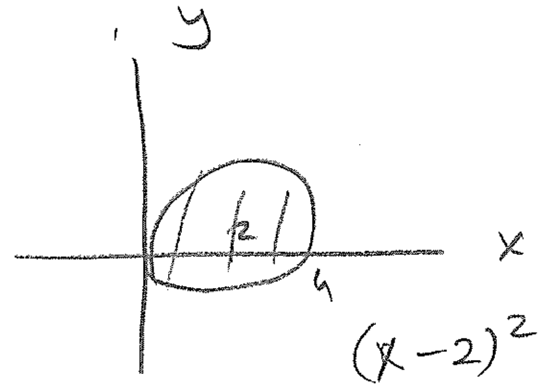
$$z = 2x^2 + y^2 = 4 - y^2 \Rightarrow \quad x^2 + y^2 = 2$$

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \\ J &= r \end{aligned}\right\} \qquad V = \iint_B \int_{2x^2+y^2}^{4-y^2} dz\,dy\,dx$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{-r^2+r^2\cos\theta}^{4-r^2\sin\theta} r \cdot dz\,dr\,d\theta = 4\pi$$

③ $x^2+y^2=4x$ silindiri, $x^2+y^2=4z$ paraboloidi ve x-y düzlemi arasında kalan bölgenin hacmi?





$$(x-2)^2+y^2=4$$

$$r=4\cos\theta$$

$$V=\iint_B \int_0^{\frac{x^2+y^2}{4}} dz\,dy\,dx=\int_{-\frac{\pi}{2}}^{\pi/2}\int_0^{4\cos\theta}\int_0^{\frac{r^2}{4}} r\,dz\,dr\,d\theta=6\pi$$